

An Improved Tripartite Bell-type Inequality

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Abstract

So far, various Bell type inequalities have been introduced to test the local realism in tripartite systems. In this article we consider a tripartite system with two measurements in each side and two outputs for each measurement. Then we will introduce a new bell type inequality for this system and we show that this inequality is violated by quantum theory with a violation factor and amount of violation of 3.5 and 2.5 respectively, which exceed those of available inequalities in both cases. Also we will show that the white noise tolerance of this new Bell type expression is 0.5, which agrees with the maximum amount of white noise tolerance of available inequalities up to now.

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I. INTRODUCTION

Using local theory, John S. Bell introduced an inequality which is violated by quantum theory. Later experiments showed that quantum theory is basically non-local [1].

As the non-locality feature of quantum theory is intensively used in quantum information, Bell type inequalities have received more attention in recent years [2].

Bell original inequality did not have any capability to be studied empirically in the laboratories. After that, Clauser, Horne, Shimony and Holt introduced their famous inequality called CHSH which was reconsidered in laboratories since then. As no experiment is error-free, there was an endeavor to gain a kind of Bell type inequality that would be violated as much as possible, so that it would be experimentally easy to test non-locality feature of quantum theory [3].

Svetlichny (in 1987) and Mermin (in 1990) obtained inequalities for tripartite systems which implied stronger violation of local theories [5, 7]. Also in 1989, Greenberger, Horn, and Zeilinger obtained some inequalities for N-Particle systems ($N > 2$) [4].

In this article, we introduce a new Bell type expression for tripartite systems with two measurements in each side and two outputs for each measurement. Then, we will show that the violation factor (i.e. the ratio of the value of Bell expression according to quantum theory to its value according to local theory) and the amount of violation (i.e. the difference between the value of Bell expression according to quantum theory and its extremum value according to local theory) of this inequality exceed those of available inequalities [5, 7, 8], while its white noise tolerance agrees with the previous results.

II. TRIPARTITE SYSTEMS

We consider a tripartite system consisting of particles \mathcal{A} , \mathcal{B} and \mathcal{C} .

Two possible measurements A and A' are performed on particle \mathcal{A} with outputs a and a' respectively, B and B' on particle \mathcal{B} with outputs b and b' respectively, and finally C and C' on particle \mathcal{C} with outcomes c and c' respectively, where $a, a', b, b', c, c' \in \{0, 1\}$.

Let $P_L(A, A', B, B', C, C'|a, a', b, b', c, c')$ denotes the triple joint probability that measurements A, A' on particle \mathcal{A} result a and a' respectively, measurements B, B' on particle \mathcal{B} result b and b' respectively, and measurements C and C' on particle \mathcal{C} result c and c'

respectively.

It is obvious that:

$$\sum_{a,a'} \sum_{b,b'} \sum_{c,c'} P_L(A, A', B, B', C, C'|a, a', b, b', c, c') = 1 \quad (1)$$

Also let $P(A, B, C|a, b, c)$ denotes the joint probability that measurement A on particle \mathcal{A} results " a ", measurement B on particle \mathcal{B} results " b ", and measurement C on particle \mathcal{C} results " c ".

Clearly:

$$P(A, B, C|a, b, c) = \sum_{a'} \sum_{b'} \sum_{c'} P_L(A, A', B, B', C, C'|a, a', b, b', c, c') \quad (2)$$

The normalization of P 's implies:

$$\sum_a \sum_b \sum_c P(A, B, C|a, b, c) = 1 \quad (3)$$

As it is well known, a Bell type expression, \mathbb{B} , is a linear combination of joint probabilities that is bounded by local theories, i.e.

$$\mathbb{B} = \sum_{I,J,K,l,m,n} \gamma(I, J, K|l, m, n) P(I, J, K|l, m, n) \quad (4)$$

where $I \in \{A, A'\}$, $J \in \{B, B'\}$, $K \in \{C, C'\}$, $l \in \{a, a'\}$, $m \in \{b, b'\}$ and $n \in \{c, c'\}$. Using equation (2) the Bell inequality in terms of P_L 's would become:

$$\mathbb{B} = \sum_{a,a'} \sum_{b,b'} \sum_{c,c'} [\alpha(a, a', b, b', c, c') - \beta(a, a', b, b', c, c')] P_L(A, A', B, B', C, C'|a, a', b, b', c, c'). \quad (5)$$

It is clear that

$$-e \leq \mathbb{B} \leq f \quad (6)$$

where f (e) is the greatest of positive real numbers α 's (β 's) in equation (5).

III. A NEW BELL EXPRESSION

One of the well-know Bell type expressions for tripartite systems is Mermin inequality, which can be expressed as [9]:

$$M = |E(A, B', C') + E(A', B, C') + E(A', B', C) - E(A, B, C)| \quad (7)$$

where

$$E(A, B, C) = \langle A, B, C \rangle = \sum_a \sum_b \sum_c (-1)^z P(A, B, C|a, b, c) \quad (8)$$

and $P(A, B, C|a, b, c)$ is the joint probability discussed above. In the above equation "z" is the number of zero's resulted in each particular setting [5].

It is shown in [7] that Mermin inequality for local theories satisfies

$$0 \leq M \leq 2 \quad (9)$$

However, according to quantum theory, the upper bound of Mermin inequality is 4 which shows that quantum theory is non-local. Here, the violation factor and amount of violation in Mermin inequality are both 2 and the maximum white noise tolerance calculated is 0.5 [9].

Now let's consider the following inequality for a tripartite system

$$\begin{aligned} G = & P(A, B, C|1, 1, 1) + 5P(A, B, C|1, 0, 0) + 5P(A, B, C|0, 0, 1) + \\ & P(A, B, C|1, 0, 1) + 4P(A, B, C|0, 0, 0) + 4P(A, B, C|0, 1, 0) + \\ & P(A, B', C'|0, 0, 0) + P(A, B', C'|0, 1, 1) - 4P(A, B', C'|0, 0, 1) - \\ & 4P(A, B', C'|0, 1, 0) - P(A', B', C|0, 0, 1) - P(A', B', C|1, 1, 1) - \\ & 4P(A', B', C|0, 1, 0) - 4P(A', B', C|1, 0, 0) - 5P(A', B, C'|1, 0, 0) - \\ & 5P(A', B, C'|0, 0, 1) + P(A', B', C'|1, 1, 0) + P(A', B', C'|0, 0, 1) - \\ & 4P(A', B', C'|1, 1, 1) - 4P(A', B', C'|0, 0, 0) \end{aligned} \quad (10)$$

In appendix A, it is shown that

$$G \leq 1. \quad (11)$$

However for a three-qubit Greenberger-Horne-Zeilinger state [4] which is

$$|\Psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle_z + |\downarrow\downarrow\downarrow\rangle_z), \quad (12)$$

where \uparrow and \downarrow are spin polarization along z axis, and if $A = \sigma_X^A$, $A' = \sigma_Y^A$, $B = \sigma_X^B$, $B' = \sigma_Y^B$, $C = \sigma_X^C$ and $C' = \sigma_Y^C$, G would become

$$G = \frac{1}{4} + \frac{5}{4} + \frac{5}{4} + 0 + 0 + \frac{4}{4} + \frac{1}{4} + \frac{1}{4} - 0 - 0 - 0 - 0 - 0 - 0 - 0 - 0 + \frac{1}{8} + \frac{1}{8} - \frac{4}{8} - \frac{4}{8} = \frac{7}{2} \quad (13)$$

It is seen that the violation factor and the amount of violation of the inequality (10) are 3.5 and 2.5 respectively, whereas the maximum violation factor and maximum amount of violation of the available inequalities so far, are 2.

To calculate the white noise tolerance of G in tripartite systems, we consider the following density matrix:

$$\rho = (1 - p)|\Psi\rangle_{GHZ} \langle\Psi| + \frac{p}{8}I. \quad (14)$$

Obviously

$$P(A, B, C|a, b, c) = \frac{p}{8} + (1 - p)P_{QM}(A, B, C|a, b, c) \quad (15)$$

where $P_{QM}(A, B, C|a, b, c)$ is the joint probability according to quantum theory and p is the tolerance of Bell type expression. From equations (10) and (15), we have:

$$p = \frac{G_{QM} - G_L}{G_{QM} - \frac{m-n}{8}} \quad (16)$$

where G_{QM} is the value of our Bell expression, G , according to quantum theory, G_L is its maximum value, according to local theories and $m(n)$ is the number of positive (negative) terms in G .

It is easily seen that the white noise tolerance of G is 0.5, which agrees with the maximum value calculated up to now.

IV. CONCLUSION

In this article we introduced a Bell type inequality for tripartite systems with two measurements for each side and two outputs for each measurement, which is violated by quantum theory with a stronger violation factor and more amount of violation than the available inequalities. In fact, the violation factor and the amount of violation of our inequality are 3.5 and 2.5 respectively, which are 1.5 and 0.5 more than the results obtained so far, respectively. However the tolerance of our inequality is the same as others. This increment of violation factor and the amount of violation increase the accuracy of experiments in which the errors are inevitable.

Also one of the advantages of our inequality is that it includes only 20 different joint probabilities whereas in other works it is much more than this (in Mermin and Svetlichny inequalities it is 32 and 64 respectively). So, our inequality requires less measurements which in turn, reduces the errors due to experiment. See [10].

Appendix A: List of Joint Probabilities

In this appendix we derive equation (11). From the definition (2) and denoting $P_L(A, A', B, B', C, C'|a, a', b, b', c, c') = P_{aa'bb'cc'}$ for simplicity, we have

$$\begin{aligned}
P(A, B, C|1, 1, 1) &= P_{101010} + P_{101011} + P_{101110} + P_{101111} + \\
&\quad P_{111010} + P_{111011} + P_{111110} + P_{111111} \\
P(A, B, C|1, 0, 0) &= P_{100000} + P_{100001} + P_{100100} + P_{100101} + \\
&\quad P_{110000} + P_{110001} + P_{110100} + P_{110101} \\
P(A, B, C|0, 0, 1) &= P_{000010} + P_{000011} + P_{000110} + P_{000111} + \\
&\quad P_{010010} + P_{010011} + P_{010110} + P_{010111} \\
P(A, B, C|1, 0, 1) &= P_{100010} + P_{100011} + P_{100110} + P_{100111} + \\
&\quad P_{110010} + P_{110011} + P_{110110} + P_{110111} \\
P(A, B, C|0, 0, 0) &= P_{000000} + P_{000001} + P_{000100} + P_{000101} + \\
&\quad P_{010000} + P_{010001} + P_{010100} + P_{010101} \\
P(A, B, C|0, 1, 0) &= P_{001000} + P_{001001} + P_{001100} + P_{001101} + \\
&\quad P_{011000} + P_{011001} + P_{011100} + P_{011101} \\
P(A, B', C'|0, 0, 0) &= P_{000000} + P_{000010} + P_{001000} + P_{001010} + \\
&\quad P_{010000} + P_{010010} + P_{011000} + P_{011010} \\
P(A, B', C'|0, 1, 1) &= P_{000101} + P_{000111} + P_{001101} + P_{001111} + \\
&\quad P_{010101} + P_{010111} + P_{011101} + P_{011111} \\
P(A, B', C'|0, 0, 1) &= P_{000001} + P_{000011} + P_{001100} + P_{001011} + \\
&\quad P_{010001} + P_{010011} + P_{011001} + P_{011011} \\
P(A, B', C'|0, 1, 0) &= P_{000100} + P_{000110} + P_{001100} + P_{001110} + \\
&\quad P_{010100} + P_{010110} + P_{011100} + P_{011110} \\
P(A', B', C|0, 0, 1) &= P_{000010} + P_{000011} + P_{001010} + P_{001011} + \\
&\quad P_{100010} + P_{100011} + P_{101010} + P_{101011} \\
P(A', B', C|1, 1, 1) &= P_{010110} + P_{010111} + P_{011110} + P_{011111} + \\
&\quad P_{110110} + P_{110111} + P_{111110} + P_{111111}
\end{aligned}$$

$$\begin{aligned}
P(A', B', C|0, 1, 0) &= P_{000100} + P_{000101} + P_{001100} + P_{001101} + \\
&\quad P_{100100} + P_{100101} + P_{101100} + P_{101101} \\
P(A', B', C|1, 0, 0) &= P_{010000} + P_{010001} + P_{011000} + P_{011001} + \\
&\quad P_{110000} + P_{110001} + P_{111000} + P_{111001} \\
P(A', B, C'|1, 0, 0) &= P_{010000} + P_{010010} + P_{010100} + P_{010110} + \\
&\quad P_{110000} + P_{110010} + P_{110100} + P_{110110} \\
P(A', B, C'|0, 0, 1) &= P_{000001} + P_{000011} + P_{000101} + P_{000111} + \\
&\quad P_{100001} + P_{100011} + P_{100101} + P_{100111} \\
P(A', B', C'|1, 1, 0) &= P_{010100} + P_{010110} + P_{011100} + P_{011110} + \\
&\quad P_{110100} + P_{110110} + P_{111100} + P_{111110} \\
P(A', B', C'|0, 0, 1) &= P_{000001} + P_{000011} + P_{101001} + P_{101011} + \\
&\quad P_{100001} + P_{100011} + P_{111110} + P_{111111} \\
P(A', B', C'|0, 0, 0) &= P_{000000} + P_{000010} + P_{001000} + P_{001010} + \\
&\quad P_{100000} + P_{100010} + P_{101000} + P_{101010} \\
P(A', B', C'|1, 1, 1) &= P_{010101} + P_{010111} + P_{011101} + P_{011111} + \\
&\quad P_{110101} + P_{110111} + P_{111101} + P_{111111}.
\end{aligned}$$

Inserting the above joint probabilities in equation (10), we get

$$\begin{aligned}
G &= P_{000000} + P_{100000} + P_{001000} - 4P_{010000} - 4P_{000100} + \\
&\quad P_{000010} - 4P_{000001} - 4P_{110000} - 4P_{101000} + P_{100100} - \\
&\quad 4P_{100010} + P_{100001} + P_{011000} - 4P_{010100} + P_{010010} - \\
&\quad 4P_{010001} - 4P_{001100} - 4P_{001010} + P_{001001} + P_{000110} - \\
&\quad 4P_{000101} - 4P_{000011} - 4P_{111000} + P_{110100} - 4P_{110010} + \\
&\quad P_{110001} + P_{011100} + P_{011010} - 4P_{011001} - 4P_{001110} + \\
&\quad P_{001101} + P_{000111} - 4P_{111111} - 4P_{011111} + P_{101111} - \\
&\quad 4P_{110111} + P_{111011} - 4P_{111101} + P_{111110} + P_{001111} + \\
&\quad P_{010111} - 4P_{011011} + P_{011101} - 4P_{011110} - 4P_{100111} + \\
&\quad P_{101011} - 4P_{101101} + P_{101110} + P_{110011} + P_{110101} -
\end{aligned}$$

$$\begin{aligned}
& 4P_{110110} - 4P_{111001} + P_{111010} + P_{111100} - 4P_{101100} - \\
& 4P_{001011} - 4P_{100011} - 4P_{100101} + P_{100110} - 4P_{101010} + \\
& P_{010101} + P_{010011} - 4P_{010110} + P_{101001},
\end{aligned}$$

which according to equation 11 is less than or equal to 1. Please note that all P's are positive here.

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